Complex Analysis: Resit Exam

Aletta Jacobshal 02, Thursday 13 April 2017, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of your exam sheet and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (30 points)

Consider the function

$$f(z) = ze^{iz}.$$

(a) (8 points) Prove that

$$f(z) = e^{-y}(x\cos x - y\sin x) + ie^{-y}(y\cos x + x\sin x),$$

where $z = x + iy$.

Solution

We compute

$$\begin{aligned} f(z) = ze^{iz} &= (x+iy)e^{i(x+iy)} = (x+iy)e^{ix-y} \\ &= e^{-y}(x+iy)(\cos x + i\sin x) \\ &= e^{-y}(x\cos x - y\sin x) + ie^{-y}(y\cos x + x\sin x). \end{aligned}$$

(b) (8 points) Prove, using the Cauchy-Riemann equations, that $f(z)$ is entire.

Solution

Let $u = e^{-y}(x\cos x - y\sin x)$ and $v = e^{-y}(y\cos x + x\sin x)$. Then

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^{-y}(\cos x - x\sin x - y\cos x), \\ \frac{\partial v}{\partial y} &= -e^{-y}(y\cos x + x\sin x) + e^{-y}\cos x = e^{-y}(-y\cos x - x\sin x + \cos x), \\ \frac{\partial u}{\partial y} &= -e^{-y}(x\cos x - y\sin x) - e^{-y}\sin x = -e^{-y}(x\cos x - y\sin x + \sin x), \\ \frac{\partial v}{\partial x} &= e^{-y}(-y\sin x + \sin x + x\cos x). \end{aligned}$$

Since all the partial derivatives are continuous for all $x + iy \in \mathbb{C}$, and the Cauchy-Riemann equations hold, that is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

for all $x + iy \in \mathbb{C}$, we conclude that the function $f(z)$ is entire.

(c) (6 points) Compute the derivative of $f(z)$.

Solution

$$f'(z) = (ze^{iz})' = e^{iz} + ize^{iz} = (1 + iz)e^{iz}.$$

(d) (8 points) Prove that the function

$$u(x, y) = e^{-y}(x \cos x - y \sin x),$$

is harmonic in $\mathbb{R}^2$ and find a harmonic conjugate of $u(x, y)$.

Solution

The given function $u(x, y)$ is harmonic in $\mathbb{R}^2$ because it is the real part of the entire function $f(z)$.

A harmonic conjugate for $u(x, y)$ is then the imaginary part of $f(z)$, that is,

$$v(x, y) = e^{-y}(y \cos x + x \sin x).$$

Question 2 (15 points)

Evaluate

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 1} dx$$

using the calculus of residues.

Solution

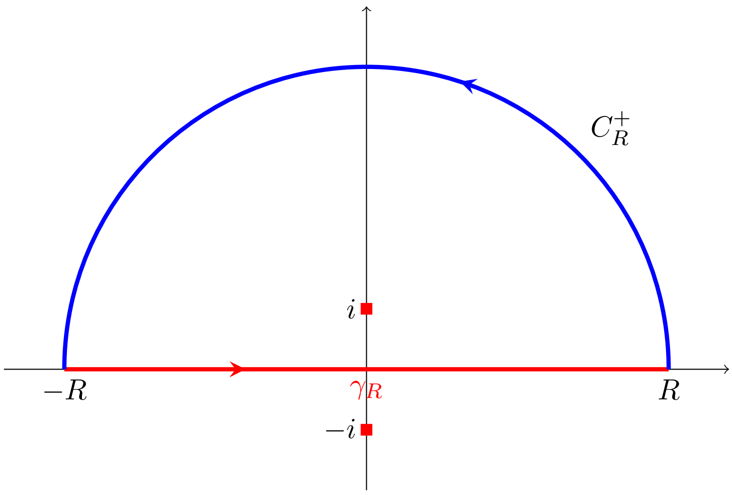
By definition,

$$\begin{aligned} I &= \text{pv} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 1} dx \\ &= \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ix}}{x^2 + 1} dx \\ &= \lim_{R \to \infty} I_R. \end{aligned}$$

To compute this integral we consider the closed contour

$$\Gamma_R = \gamma_R + C_R^+,$$

shown below.



We have

$$I_R = \int_{-R}^{R} \frac{e^{ix}}{x^2+1}dx$$
$$= \int_{\gamma_R} f(z)dz,$$

where

$$f(z) = \frac{e^{iz}}{z^2+1}.$$

Therefore,

$$\int_{\Gamma_R} f(z)dz = I_R + \int_{C_R^+} f(z)dz.$$

For $R > 1$ we have

$$\int_{\Gamma_R} f(z)dz = 2\pi i \operatorname{Res}(i) = \frac{\pi}{e},$$

where we used that

$$\operatorname{Res}(i) = \lim_{z\to i}(z-i)\frac{e^{iz}}{(z-i)(z+i)} = \lim_{z\to i}\frac{e^{iz}}{(z+i)} = \frac{e^{-1}}{2i} = \frac{1}{2ie}.$$

Moreover, since the degree of the denominator is 2 and we have an expression of the form e^{iz} in the numerator we can apply Jordan's lemma for C_R^+ to get

$$\lim_{R\to\infty}\int_{C_R^+} f(z)dz = 0.$$

Then taking the limit $R \to \infty$ we get

$$\frac{\pi}{e} = I + 0,$$

giving

$$I = \frac{\pi}{e}.$$

Question 3 (10 points)

Use Rouché's theorem to show that, if $0 < \varepsilon < 7/4$, then the polynomial $P(z) = z^3 + \varepsilon z^2 - 1$ has exactly 3 roots in the disk $|z| < 2$.

Solution

The functions $f(z) = z^3 - 1$ and $h(z) = \varepsilon z^2$ are both analytic on and inside the circle $|z| = 2$. The number of zeros of $f(z) = z^3 - 1$ inside the disk $|z| < 2$, counting multiplicity, is $N_0(f) = 3$. Moreover, on the circle $|z| = 2$ we have

$$|h(z)| = \varepsilon |z|^2 = 4\varepsilon,$$

and

$$|f(z)| = |z^3 - 1| \geq ||z^3| - 1| = 7.$$

Therefore, we can apply Rouché's theorem when $4\varepsilon < 7$, which implies $|h(z)| < |f(z)|$, to get for $P(z) = f(z) + h(z)$ that the number of its roots inside the disk $|z| < 2$ is

$$N_0(P) = N_0(f) = 3.$$

Question 4 (15 points)

Represent the function

$$f(z) = \frac{z}{z^2 - 1},$$

(a) (8 points) as a Taylor series around 0 and find its radius of convergence;

Solution

$$\begin{aligned} \frac{z}{z^2 - 1} &= -z(1 + z^2 + z^4 + z^6 + z^8 + \cdots) \\ &= -z - z^3 - z^5 - z^7 - z^9 \cdots, \end{aligned}$$

where we used the geometric series for $1/(1 - z^2)$. The geometric series converges when $|z^2| < 1$, that is, for $|z| < 1$. Therefore, we conclude that the radius of convergence must be 1.

Alternatively, the function $f(z)$ has singularities at $z = \pm 1$ which are both at a distance $|z| = 1$ from 0. Therefore, the radius of convergence is 1.

(b) (7 points) as a Laurent series in the domain $|z| > 1$.

Solution

Since $|z| > 1$, that is $|1/z^2| < 1$, we have

$$\frac{z}{z^2 - 1} = \frac{\frac{1}{z}}{1 - \frac{1}{z^2}} = \frac{1}{z}\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots\right).$$

Therefore, for $|z| > 1$ we can write

$$\frac{z}{z^2 - 1} = \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \frac{1}{z^7} + \cdots$$

Question 5 (10 points)

Consider the functions

$$f(z) = \frac{\sin z}{z} \quad \text{and} \quad g(z) = e^{1/z}.$$

Determine the singularities of $f(z)$ and $g(z)$, and their types (removable, pole, essential; if pole, specify the order). Make sure to justify your answer.

Solution

The function $f(z)$ is singular at $z = 0$. The Laurent series for $|z| > 0$ is given by

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \cdots\right) = 1 - \frac{z^2}{3!} + \cdots .$$

Since there are no negative powers we conclude that $z = 0$ is a removable singularity.

The function $g(z)$ is singular at $z = 0$. The Laurent series for $|z| > 0$ is given by

$$g(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \cdots .$$

Since there are infinitely many negative powers we conclude that $z = 0$ is an essential singularity.

Question 6 (10 points)

Consider a function $f(z)$ such that $\mathrm{Re}(f(z)) \geq M$ for all $z \in \mathbb{C}$, where M is a real constant. Prove that if $f(z)$ is entire then it must be constant. *Hint: consider the function* $e^{-f(z)}$.

Solution

Let

$$g(z) = e^{-f(z)}.$$

If $f(z)$ is entire, then so is $g(z)$. Moreover, if we write $f = u + iv$, with $u = \mathrm{Re}(f(z))$, $v = \mathrm{Im}(f(z))$, then we have

$$|g(z)| = |e^{-f(z)}| = |e^{-u-iv}| = |e^{-u}| \leq e^{-M}.$$

Since $g(z)$ is a bounded entire function we conclude from Liouville's theorem that $g(z)$ is constant $c \in \mathbb{C}$.

Therefore, $e^{-f(z)} = c$. This implies $f(z) = -\operatorname{Log} c + 2k(z)\pi i$, with $k(z)$ a $\mathbb{Z}$-valued function. Since $f(z)$ is continuous (being entire) we conclude that $k(z)$ is also continuous. The only continuous functions from $\mathbb{C}$ to $\mathbb{Z}$ are constant functions, therefore $k(z) = K$. Then $f(z) = -\operatorname{Log} c + 2K\pi i$ is a constant function.